1.1 (a) *For the sequence*

$$1.1, -1.01, 1.001, -1.0001, 1.00001, -1.000001, \cdots$$

find the limit inferior, limit superior, infimum and the supremum.

Solution: Note that $a_n = 1 + \frac{1}{10^n}$ if n is odd and $a_n = -(1 + \frac{1}{10^n})$ if n is even. We see that $a_{2n} \to -1$ and $a_{2n+1} \to 1$ as $n \to \infty$. Thus, $\liminf a_n = -1$ and $\limsup a_n = 1$. Since $1/10^n$ is decreasing, it is trivial to see that $\inf a_n = -1.01$ and $\sup a_n = 1.1$.

1.1 (b) *If* $\{a_n\}$ *is a sequence of positive, real numbers such that the* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, *then prove that* $\lim_{n\to\infty} a_n^{1/n} = l$.

Solution: Theorem 3.37 of Priniciples of Mathematical Analysis by Walter Rudin tells that if $\{a_n\}$ is a sequence of positive, real numbers such that the sequence a_{n+1}/a_n converges, then $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} a_n^{1/n}$.

1.1 (c) *By considering the sequence* $1, a, ab, a^2b, a^2b^2, a^3b^2, a^3b^3, \cdots$ *where* a, b *are distinct positive numbers, show that the converse of (b) is not true.*

Solution: Observe that $x_{2n} = (ab)^n$ and $x_{2n+1} = a\, x_{2n}$. Thus, $x_{2n}^{1/2n} = (ab)^{n/2n} \to \sqrt{ab}$ as $n \to \infty$ and $(x_{2n+1})^{1/(2n+1)} = a^{1/(2n+1)}\,(ab)^{n/(2n+1)} \to \sqrt{ab}$ as $n \to \infty$. Therefore, $\lim\limits_{n\to\infty} x_n^{1/n} = \sqrt{ab}$. If n is even ($n = 2k$ for some k), then

$$\frac{x_{n+1}}{x_n} = \frac{x_{2k+1}}{x_{2k}} = \frac{a(ab)^k}{(ab)^k} \to a \text{ as } n \to \infty.$$

If n is odd ($n = 2k + 1$ for some k), then

$$\frac{x_{n+1}}{x_n} = \frac{x_{2k+2}}{x_{2k+1}} = \frac{(ab)^{k+1}}{a(ab)^k} \to b \text{ as } n \to \infty.$$

Therefore, $\lim \frac{x_{n+1}}{x_n}$ does not exist as $a \neq b$.

1.2 (a) *Let* a, b *be real numbers with* $a > 0$. *Prove that the infimum of the set* $\{an+b/n : n \in \mathbb{N}\}$ *equals* $a+b$ *if* $b \le 2a$ *and equals* $am + b/m$ *when* $b > 2a$, *where* $m = min\{k \in \mathbb{N} : k \ge -1/2 + \sqrt{b/a + 1/4}\}$.

Solution: Set $x_n = an + b/n$. Then,

$$x_{n+1} - x_n = a + b\left(\frac{1}{n+1} - \frac{1}{n}\right) = a - \frac{b}{n(n+1)}.$$

Case- I: ($b \le 2a$)
For $b \le 2a$, we have

$$\frac{b}{n(n+1)} \le \frac{b}{2} \le a \text{ (because } \frac{1}{n(n+1)} \le \frac{1}{2}) \text{ for all } n.$$

Therefore, $x_{n+1} - x_n \geq 0$. That is, $\{x_n\}$ is an increasing sequence. Hence,

$$\inf\{x_n : n \in \mathbb{N}\} = x_1 = a + b.$$

Case-II: $(b > 2a)$
Set $m = \min\{k : k(k+1) \geq b/a\}$. Note that $k(k+1) \geq b/a \Leftrightarrow (k+1/2)^2 \geq (b/a + 1/4) \Leftrightarrow |k+1/2| \geq \sqrt{(b/a+1/4)}$. That is $k \geq -1/2 + \sqrt{(b/a+1/4)}$. For $n < m$, we have $n(n+1) < b/a$, which implies that $x_{n+1} - x_n < 0$. That is,

$$x_1 > x_2 > x_3 > \cdots x_m.$$

For $n \geq m$, we have $n(n+1) \geq b/a$, which implies that $x_{n+1} - x_n \geq 0$. That is,

$$x_m \leq x_{m+1} \leq x_{m+2} \leq \cdots.$$

Therefore,

$$\inf\{x_n : n \in \mathbb{N}\} = x_m = am + \frac{b}{m}.$$

1.2 (b) *For any real t, prove that $\lim_{n\to\infty} \frac{t^n}{n!} = 0$.*

Solution: Let $x_n = \frac{t^n}{n!}$. Then,

$$\frac{x_{n+1}}{x_n} = \frac{t}{n+1} \to 0 \text{ as } n \to \infty.$$

Therefore, by Theorem 3.2.11 of Introduction to Real Analysis by Robert G. Bartle and Donald R. Sherbert, we can conclude that $\lim_{n\to\infty} \frac{t^n}{n!} = 0$ for all real t. $\square$

2.1 (a) *Test the convergence of the series $\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n}}$.*

Solution: Since $\{\frac{1}{\sqrt{n}}\}$ is a decreasing sequence of positive numbers with $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, by Theorem 9.3.2 of Introduction to Real Analysis by Robert G. Bartle and Donald R. Sherbert, we can conclude that $\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n}}$ converges.

2.1 (b) *Let $\{a_n\}$ be a sequence of non-zero real numbers. Assume*

$$\lim_{n\to\infty} n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right)$$

exists and is > 1. Prove that $\sum_n a_n$ converges absolutely.

Solution:

Since $\lim_{n\to\infty} n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right)$ exists, we can conclude that $\left|\frac{a_n}{a_{n+1}}\right| \to 1$. Therefore,

$$\lim_{n\to\infty} n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right) = \lim_{n\to\infty} n\left(1 - \left|\frac{a_{n+1}}{a_n}\right|\right).$$

By Corollary 9.2.9 of Introduction to Real Analysis by Robert G. Bartle and Donald R. Sherbert, we can deduce that $\sum_n a_n$ converges absolutely.

2.2 (a) *If $\sum_{n\geq 1} a_n$ is an absolutely convergent series of real numbers, and σ is a bijection of the set of natural numbers to itself, prove that $\sum_{n\geq 1} a_{\sigma(n)}$ also converges to the same sum.*

Solution: See Theorem 9.1.5, Introduction to Real Analysis by Robert G. Bartle and Donald R. Sherbert

2.2 (b) *Prove that the series $\sum_{n\geq 0} \frac{1}{n!+(n+1)!}$ converges to 1.*

Solution: By rewriting

$$\frac{1}{n!+(n+1)!} = \frac{1}{n!(n+2)} = \frac{n+1}{(n+2)!} = \frac{1}{(n+1)!} - \frac{1}{(n+2)!},$$

we see that

$$\begin{aligned}\sum_{n=0}^{N} \frac{1}{n!+(n+1)!} &= \sum_{n=0}^{N}\left(\frac{1}{(n+1)!} - \frac{1}{(n+2)!}\right)\\ &= \left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \cdots \left(\frac{1}{(N+1)!} - \frac{1}{(N+2)!}\right)\\ &= 1 - \frac{1}{(N+2)!}.\end{aligned}$$

Since $\lim\limits_{N\to\infty} \sum\limits_{n=0}^{N} \frac{1}{n!+(n+1)!} = \lim\limits_{N\to\infty}\left(1 - \frac{1}{(N+2)!}\right) = 1$, we get $\sum\limits_{n=0}^{\infty} \frac{1}{n!+(n+1)!} = 1$.

□

3 (a) *Let S be a subset of $\mathbb{R}$. Define its interior S^0.*
Prove that $(S^0)^c = \overline{S^c}$, where A^c denotes the complement of a set A.

Solution: Interior A^0 of a set A is the largest open set contained in A. Since $S^0 \subseteq S$, we have $S^c \subseteq (S^0)^c =$ closed set (since S^0 is open). Thus, $\overline{S^c} \subseteq (S^0)^c$ and therefore $S^0 \subseteq \overline{(S^c)}^c$.

Other way inclusion: Since $\overline{(S^c)}$ is a closed set containing S^c, $\overline{(S^c)}^c$ is a open set contained in S. Thus, $\overline{(S^c)}^c \subseteq S^0$. Hence we proved that $S^0 = \overline{(S^c)}^c$. That is, $(S^0)^c = \overline{S^c}$.

3 (b) *Let S be a subset of $\mathbb{R}$. Define its closure $\bar{S}$.*
Prove that $(\overline{S})^c = (S^c)^0$.

Solution: Closure $\bar{A}$ of a set A is the smallest closed set that contains A. From the question $3(a)$, we have $S^0 = \overline{(S^c)}^c$. By replacing S by S^c, we get $(\overline{S})^c = (S^c)^0$. Hence proved.

□

4.1 (a) *For the function $f(x) = \frac{1}{e^{1/x}+1}$ defined for $x \neq 0$, determine whether the left hand and right hand limits exist at 0. Draw a rough graph of $f(x)$.*

Solution: Let us consider the function $f(x) = \frac{1}{e^{1/x}+1}$. When $x \to 0+, 1/x \to \infty$. Thus, $e^{1/x} \to \infty$ and therefore $\frac{1}{e^{1/x}+1} \to 0$. i.e.,

$$\lim_{x\to 0+} f(x) = 0.$$

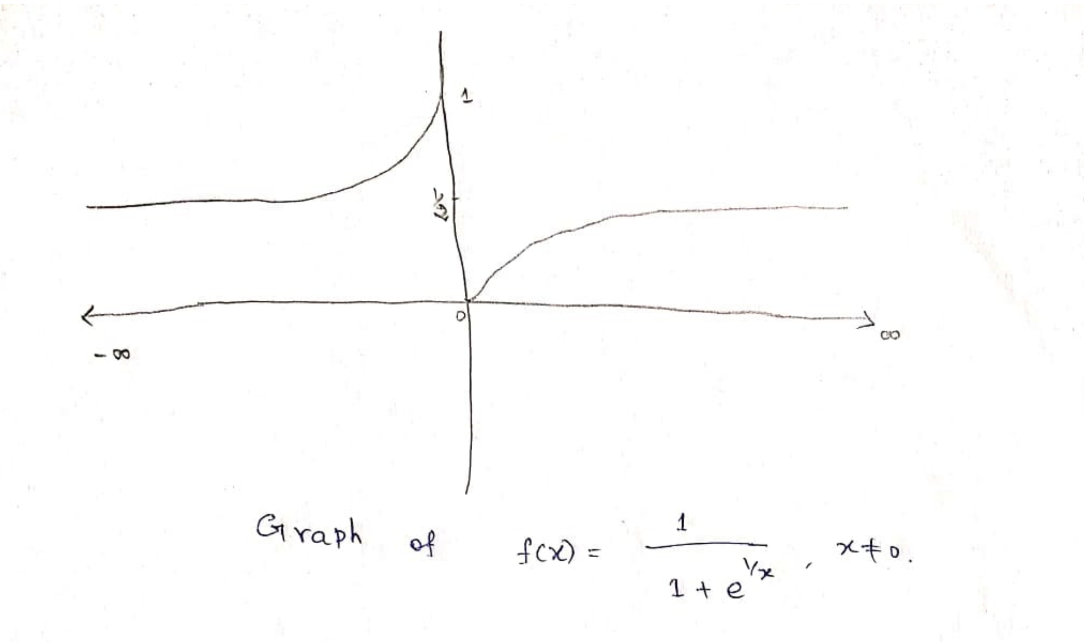

Figure 1: A rough graph of $f(x)$

When $x \to 0-, 1/x \to -\infty$. Thus, $e^{1/x} \to 0$ and therefore $\frac{1}{e^{1/x}+1} \to 1$. i.e.,

$$\lim_{x \to 0-} f(x) = 1.$$

4.1 (b) *Prove that a uniformly continuous function defined on a bounded subset of* $\mathbb{R}$ *must be bounded.*

Solution: Suppose $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a uniformly continuous and A is bounded. Then, by Theorem 5.4.8 of Introduction to Real Analysis by Robert G. Bartle and Donald R. Sherbert, f has a continuous extension g on $\overline{A}$. That is, $g : \overline{A} \to \mathbb{R}$ is a continuous function and $f(x) = g(x)$ for all $x \in A$. Since A is bounded, $\overline{A}$ is compact and hence g is a bounded function. Therefore, f is bounded.

4.2 (a) *Prove that there exists no continuous bijection f from* $(0, 1)$ *to* $[0, 1]$.

Solution: Suppose $f : (0, 1) \to [0, 1]$ is a bijective continuous. Choose $a, b \in (0, 1)$ such that $f(a) = 0, f(b) = 1$. Without loss of generality, assume that $a < b$. Then, $f : [a, b] \to [0, 1]$ is continuous. By Intermediate Value Theorem, $f([a, b]) \supseteq [0, 1]$. Therefore $f([a, b]) = [0, 1]$ and thus $f : (0, 1) \to [0, 1]$ cannot be bijective. Hence the result follows.

4.2 (b) *Prove that the only functions $g : \mathbb{R} \to \mathbb{R}$ satisfying $|g(x) - g(y)| \leq |x - y|^2$ for all x, y are the constant functions.*

Solution: For $x \neq y$, we have

$$\left| \frac{g(x) - g(y)}{x - y} \right| \leq |x - y|.$$

By letting $y \to x$, we get that

$$g'(x) = \lim_{y \to x} \frac{g(x) - g(y)}{x - y} = 0.$$

Since $g'(x) = 0$ for all x, g must be constant.

$\square$

5.1 (a) *Compute* $lim_{x \to 0+} \frac{\log(x)}{x}$.

Solution: Since $1/x \to \infty$ and $\log x \to -\infty$ when $x \to 0+$, we get

$$\lim_{x \to 0+} \frac{1}{x} \log x = -\infty.$$

5.1 (b) *Prove that the Taylor series of* $e^x + e^{-x}$ *converges to it for all real* x.

Solution: It is trivial to see that Taylor series of $e^x = \sum \frac{x^n}{n!}$. Therefore, Taylor series of

$$e^x + e^{-x} = \sum \frac{1 + (-1)^n}{n!} x^n = \sum_{n=1}^{\infty} \frac{1}{(2n)!} y^n,$$

where $y = x^2$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{(2n)!}{(2n+2)!} = \frac{1}{(2n+1)(2n+2)} \to 0 \text{ as } n \to \infty.$$

Therefore radius of convergence of $\sum a_n y^n$ is ∞. That is, Taylor series of $e^x + e^{-x}$ converges for all real x.

5.2 (a) *Compute* $\lim_{x \to \pi/2} \frac{tan(x)}{tan(3x)}$.

Solution:

$$\begin{aligned}
\lim_{x \to \frac{\pi}{2}} \frac{\sin x}{\sin 3x} \frac{\cos 3x}{\cos x} &= -\lim_{x \to \frac{\pi}{2}} \frac{\cos 3x}{\cos x} \text{ (since } \sin(\pi/2) = 1 \text{ and } \sin(3\pi/2) = -1) \\
&= -\lim_{x \to \frac{\pi}{2}} \frac{-3 \sin 3x}{-\sin x} \text{ (by L'Hospital rule)} \\
&= 3.
\end{aligned}$$

5.2 (b) *Let* f *be a thrice differentiable function such that* $f^{(3)}$ *is continuous in a neighbourhood of* 0*. Suppose* $f(0) = f'(0) = f"(0) = 0$ *and* $f^{(3)}(0) \neq 0$*. Use Taylor's formula to deduce that* f *does not have a local extremum at* 0*.*

Solution: Since $f^{(3)}$ is continuous at 0 and $f^{(3)}(0) \neq 0$, choose $r > 0$ such that $f^{(3)}(x) \neq 0$ for all $x \in I = (-r, r)$. By Taylor's formula, for $x \in I$, we have $f(x) = \frac{f^{(3)}(a)}{3!} x^3$ for some a between 0 and x. Since f takes both positive and negative values in I and $f(0) = 0, f$ dose not have a local extremum at 0.

$\square$

6.1 (a) *Let $f : [0,1] \to \mathbb{R}$ be thrice differentiable. Suppose $f(0) = f(1) = f'(0) = f'(1) = 0$. Prove that $f^{(3)}(t) = 0$ for some $t \in (0,1)$.*

Solution: Since $f(0) = f(1) = 0$, by Mean Value Theorem, there exists a $a \in (0,1)$ such that $f'(a) = 0$. Since $f'(0) = f'(a) = 0$, there exists a $b \in (0,a)$ such that $f''(b) = 0$. Since $f'(a) = f'(1) = 0$, there exists a $c \in (a,1)$ such that $f''(c) = 0$. Since $f''(b) = f''(c) = 0$, there exists a $d \in (b,c) \subseteq (0,1)$ such that $f'''(d) = 0$. This proves the result.

6.1 (b) *Let f be an infinitely differentiable function defined on $\mathbb{R}$. Suppose $f(1/n) = 0$ for all natural numbers n. Prove that $f^{(k)}(0) = 0$ for all $k \geq 0$.*

Solution: Since $f(1/n) = 0$ for all n, continuity of f gives that $f(0) = 0$. Also, by Mean Value Theorem, for all n, there exists a $x_{1,n}$ such that $\frac{1}{n+1} < x_{1,n} < \frac{1}{n}$ and $f'(x_{1,n}) = 0$. Thus, it is easy to see that $x_{1,n} \to 0$ as $n \to \infty$ and by continuity of f', we get $f'(0) = 0$. Again by Mean Value Theorem, we can get $x_{2,n} \in (x_{1,n+1}, x_{1,n})$ such that $f''(x_{2,n}) = 0$. As $x_{2,n} \to 0$ as $n \to \infty$, by continuity of f'', we get $f''(0) = 0$. By repeating this arguments again and again, we easily see that $f^{(k)}(0) = 0$ for all $k \geq 0$.

6.2 (a) *Consider $f(x) = 2x^4 + x^4 \sin(1/x)$ for $x \neq 0$; $f(0) = 0$. Prove that in each interval $(-t,t)$, the derivative f' takes both positive and negative values.*

Solution: Take $f(x) = 2x^4 + x^4 \sin(1/x)$ for $x \neq 0$. Then, $f'(x) = 8x^3 + 4x^3 \sin(1/x) - x^2 \cos(1/x)$ for $x \neq 0$. If $1/x = -2n\pi$, then $f'(x) = x^2(8x - \cos(1/x)) = x^2(8x - 1) < 0$ as $x < 0$. If $1/x = 2n\pi + \pi/2$, then $f'(x) = 8x^3 + 4x^3 > 0$ as $x > 0$. Thus, in each interval $(-t,t)$, the derivative f' takes both positive and negative values.

6.2 (b) *Suppose g is continuous on $[0,2]$ and differentiable on $(0,2)$. If $g(0) = 0$ and $g(1) = g(2) = 1$, prove that there exists $a \in (0,2)$ such that $g'(a) = 1/2$.*

Solution:

By Mean Value Theorem, there exists a $a \in (0,2)$ such that

$$g'(a) = \frac{g(2) - g(0)}{2 - 0} = \frac{1}{2}.$$

Hence, the desired result follows. □